LOGICAL ASPECTS OF TOPOFRAMES

MOHAMMAD ZARGHANI [1], ALI AKBAR ESTAJI,
AND ABOLGHASEM KARIMI FEIZABADI

Abstract. In this paper, we prepare a general account of the role of topoframes as the Lindenbaum algebras of a certain type of propositional theories.

1. Introduction and background

Frames are complete lattices in which infinite joins distribute over finite meets. Just as Boolean logics can be seen as models for classical propositional logic, frames can be seen as models for geometric propositional logic, which is a logic with finite conjunctions and infinite disjunctions. In this paper we define a bi-system logic (L, Th) all of the elements belonging to Th have negations in L.

Before starting the main ideas, it will be helpful to make some preliminary observations.

Definition 1.1. ([2], [5]) A topoframe is a pair (L, τ), abbreviated L_τ; consisting of a frame $(L; \vee, \wedge, \perp, \top)$ and a subframe τ of L all of whose elements are complemented in L.

Definition 1.2. Let τ_i be a topoframe on a frame L_i, for every $i = 1, 2$. A frame homomorphism $f : L_1 \to L_2$ is called a (τ_1, τ_2)-*homomorphism* if $f(\tau_1) \subseteq \tau_2$.

[1] The following definition illuminate logical aspects of frames.

Definition 1.3. A propositional theory Th in this context is determined by the following ingredients.

(1) Its propositions are generated from a given set of basic propositions by binary conjunction $\wedge$ and disjunction $\vee$ of arbitrary sets of propositions, together with the distinguished propositions $\top$ ("true") and $\perp$ ("false").

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(2) The notion of entailment $\vdash$ between propositions is subject to the following rules, where $a, b, \ldots$ stand for arbitrary propositions and S for any set of propositions.
(i) $a \vdash a$.
(ii) If $a \vdash b$ and $b \vdash c$, then $a \vdash c$.
(iii) $\bigvee S \vdash a$ whenever $s \vdash a$ for all $s \in S$.
(iv) If $a \vdash b$ and $a \vdash c$, then $a \vdash b \wedge c$.
(v) $a \wedge b \vdash b$ and $a \wedge b \vdash b$.
(vi) $s \vdash \bigvee S$ for all $s \in S$.
(vii) $\bot \vdash \bigvee \emptyset$.
($viii$) $a \vdash \top$.
(ix) $a \wedge \bigvee S \vdash \bigvee\{a \wedge s \mid s \in S\}$.

Remark 1.4. We list a few simple consequences of the rules above, where "≡" expresses mutual entailment:
$a \wedge a \equiv a$, $a \wedge b \equiv b\wedge$, $a \wedge (b \wedge c) \equiv (a \wedge b) \wedge c$, $\bot \vdash a$;
if $a \vdash c$ and $b \vdash d$, then $a \wedge b \vdash c \wedge d$;
$t \wedge \bigvee S \equiv t$ for all $t \in S$;
$a \wedge \bigvee S \equiv \bigvee\{a \wedge s \mid s \in S\}$.

Next, as for other logical systems, any propositional theory Th of this kind gives rise to its Lindenbaum algebra $\Lambda(Th)$ which consists of all propositions of Th modulo provable equivalence $a \equiv b$, with operations determined by the propositional operations $\wedge$, $\bigvee$, $\top$ and $\bot$.

2. topo-propositional theories

In this section we extend well-known results for topoframes to the setting of topo-propositional, the logical bi-structures of topology and define some arrows on them yielding a closure operator and a kernel operator.

The notion of negation $\neg$ for propositions has the following rule, where a, b stand for arbitrary propositions:
if $a \vee b \equiv \top$ and $a \wedge b \equiv \bot$, then $a \equiv \neg b$

Definition 2.1. Let (L, Th) be a topo-propositional theory and Th a subset of L being also a propositional theory. The pair (L, Th) is said to be a **topo-propositional theory**, if

(1) the members of Th have negations in L, and
(2) $\Lambda(Th)$ with the propositional operations $\wedge$, $\bigvee$, $\top$ and $\bot$ is a subframe of $\Lambda(L)$.

We can immediately conclude from Definition 2.1 that for any topo-propositional theory (L, Th), its Lindenbaum algebra $(\Lambda(L), \Lambda(Th))$ is a topoframe that its inequalities determined by its axioms.

Lemma 2.2. *In a propositional theory L, for any $a, b \in L$, the following statements are equivalent:*

(1) $a \wedge b \equiv a$;

(2) $a \vdash b$;

(3) $a \vee b \equiv b$.

Proof. (1⇒2) Let $a \wedge b \equiv a$. Then $a \vdash a \wedge b$; also by 1.3(v), $a \wedge b \vdash b$ and hence $a \vdash b$, by 1.3(ii).
(2⇒3) Let $a \vdash b$; furthermore, $b \vdash b$. So $a \vee b \vdash b$, by 1.3(iii); on the other hand, $b \vdash a \vee b$ by 1.3(vi) and hence $a \vee b \equiv b$, by definition.
(3⇒2) Let $a \vee b \equiv b$; so that $a \vee b \vdash b$, by definition; furthermore, $a \vdash a \vee b$ by 1.3(vi) and hence $a \vdash b$, by transitivity law.
(2⇒1) Let $a \vdash b$; furthermore, $a \vdash a$. So $a \vdash a \wedge b$, by 1.3(iv); on the other hand, $a \wedge b \vdash a$ by 1.3(v) and hence $a \wedge b \equiv a$, by definition. □

As an immediate consequence of the latter lemma, a topo-propositional theory (L, Th) satisfies
Unit rules: $a \vee \bot \equiv a$, $a \wedge \top \equiv a$,
for every $a \in L$.

Definition 2.3. In a propositional theory L, the **pseudo-negation** of an element $a \in L$ is defined by

$$a^* := \bigvee \{t \in L \mid t \wedge a \equiv \bot\},$$

Proposition 2.4. *Let L be a propositional theory. Then, for every $a, b, c \in L$ and $\{a_i\}_i \subseteq L$, the following statements hold.*

(1) *If $a \wedge b \equiv \bot$, then $a \vdash b^*$.*
(2) *If $a \vdash b$, then $b^* \vdash a^*$.*
(3) $a \vdash a^{**}$.
(4) $(a \vee b)^* \equiv a^* \wedge b^*$.
(5) *If $a \in L$ has a negation, then $a^* \equiv \neg a$.*
(6) $a \wedge (a \vee b) \equiv a$ *and* $a \vee (a \wedge b) \equiv a$.
(7) $a \vee (b \wedge c) \equiv (a \vee b) \wedge (a \vee c)$.

In a topo-propositional theory, infinitary infima in its Lindenbaum algebra is not the same as conjunction unless we add this condition to the assumptions. For instance, see part (5) of the following proposition.

Proposition 2.5. *Let (L, Th) be a topo-propositional theory. Then, for $a, b \in Th$ and $\{a_i\}_i \subseteq L$, the following statements hold.*

(1) $\neg\neg a \equiv a$.
(2) $\neg(a \wedge b) \equiv \neg a \vee \neg b$.
(3) $\neg(a \vee b) \equiv \neg a \wedge \neg b$.

(4) *If* $a \vdash b$*, then* $\neg b \vdash \neg a$*.*

(5) *If for every* $S \subseteq \neg Th$ *the conjunction* $\bigwedge S$ *belongs to* $\neg Th$ *and* $\bigwedge S \vdash s$ *for all* $s \in S$*, then*

$$\neg(\bigvee_{i\in I} a_i) \equiv \bigwedge_{i\in I} \neg a_i \quad \text{and } \neg a \vee \bigwedge_{i\in I} \neg a_i \equiv \bigwedge_{i\in I} (\neg a \vee \neg a_i)\,.$$

Proof. (1) This is obvious, by definition.

(2) By distributivity, associativity and unit laws in L, we have

$$\begin{aligned}(a \wedge b) \wedge (\neg a \vee \neg b) &\equiv a \wedge ((b \wedge \neg a) \vee (b \wedge \neg b))\\ &\equiv a \wedge ((b \wedge \neg a) \vee \bot)\\ &\equiv a \wedge (b \wedge \neg a)\\ &\equiv \bot\end{aligned}$$

and

$$\begin{aligned}(a \wedge b) \vee (\neg a \vee \neg b) &\equiv ((a \vee \neg a) \wedge (b \vee \neg a)) \vee \neg b\\ &\equiv (\top \wedge (b \vee \neg a)) \vee \neg b\\ &\equiv (b \vee \neg a) \vee \neg b\\ &\equiv \top.\end{aligned}$$

Hence $\neg(a \wedge b) \equiv \neg a \vee \neg b$.

(3) Since every $a \in Th$ has a negation in L, by part (5) of Proposition 2.4, we have $a^* \equiv \neg a$ and so this a particular case of part (4) of Proposition 2.4.

(4) This a particular case of part (2) of Proposition 2.4.

(5) Since $a_i \vdash \bigvee_{i\in I} a_i$, by part (2), $\neg(\bigvee_{i\in I} a_i) \vdash \neg a_i$, for every i. Now, let $c \in L$ with $c \vdash \neg a_i$. Then for every $i \in I$, $a_i \vdash c^*$. It follows that $\bigvee_{i\in I} a_i \vdash c^*$. So $c \vdash c^{**} \vdash \neg(\bigvee_{i\in I} a_i)$. In particular, for $c := \bigwedge_{i\in I} \neg a_i$, we have $c \vdash \neg(\bigvee_{i\in I} a_i)$, and hence $\neg(\bigvee_{i\in I} a_i) \equiv \bigwedge_{i\in I} \neg a_i$. The second assertion follows from part (2) and the first assertion.

□

Definition 2.6. In a topo-propositional theory (L, Th), the binary relation $\rightarrow$ on $L \times L$ is defined by

$$(a \rightarrow b) := \bigvee\{t \in Th \mid t \wedge a \vdash b\}\,,$$

and if for every $S \subseteq \neg Th$, $\bigwedge S \vdash s$ for all $s \in S$, then the binary relation $\searrow$ is defined by

$$(a \searrow b) := \bigwedge\{\neg q \mid q \in Th, b \vdash a \vee q\}\,.$$

The arrows mentioned above yield the following notions.
whenever for any $S \subseteq \neg Th$ the conjunction $\bigwedge S$ belongs to $\neg Th$ and $\bigwedge S \vdash$

s for all $s \in S$, the **closure** of p in $L$is the element

$$\lozenge p := (\bot \setminus p) \equiv \bigwedge\{s \in \neg Th \mid p \vdash s\},$$

where $\neg Th := \{\neg Th \mid t \in Th\}$, and the **interior** of any $p \in L$ is the element

$$\square p := (\top \to p) \equiv \bigvee\{t \in Th \mid t \vdash p\}.$$

The properties of interior and closure are summarized in the following proposition. It says that $\square$ on L is a kernel operator and $\lozenge$ on L is a closure operator.

Proposition 2.7. *Let (L, Th) be a topo-propositional theory. Then the operation $\square$ from L to L satisfies the following properties.*

(K1) $\square\top \equiv \top$, $\square\bot \equiv \bot$.
(K2) $\square p \vdash p$.
(K3) *If $a \in Th$, then $a \equiv \square a$, and $\square p \equiv \square\square p$.*
(K4) $\square(p \wedge q) \equiv \square p \wedge \square q$,

for all $p, q \in L$. Also, whenever for any $S \subseteq \neg Th$ the conjunction $\bigwedge S$ belongs to $\neg Th$ and $\bigwedge S \vdash s$ for all $s \in S$, the operation $\lozenge$ from L to L satisfies the following properties:

(C1) $\lozenge\bot \equiv \bot$, $\lozenge\bot \equiv \bot$.
(C2) $p \vdash \lozenge p$.
(C3) $\lozenge\lozenge p \equiv \lozenge p$.
(C4) $\lozenge(p \vee q) \equiv \lozenge p \vee \lozenge q$,

for all $p, q \in L$.

Proof. (K1) By statement (*viii*) of Definition 1.3, we have $\square\top \vdash \top$. On the other hand,

$$\begin{aligned} \top &\vdash \bigvee Th && \text{by statement } (vi) \text{ of Definition 1.3} \\ &= \bigvee\{t \in Th \mid t \vdash \top\} \\ &= \square\top. \end{aligned}$$

So $\square\top \equiv \top$. The other assertion is easy to verify.

(K2) Let $S := \{t \in Th \mid t \vdash p\}$. Then $\square p \equiv \bigvee S \vdash p$, by statement (*vi*) of Definition 1.3.

(K3) For $a \in Th$, let $S := \{t \in Th \mid t \vdash a\}$. Then, by statement (*vi*) of Definition 1.3, $a \vdash \bigvee S = \square a$, since a belongs to S. Furthermore, by assertion (K2), $\square a \vdash a$, and hence $a \equiv \square a$. The second assertion follows from the fact that $\square p$ belongs to S.

(K4) By assertion (K2), $\square p \wedge \square q \vdash p \wedge q$ and hence $\square(\square p \wedge \square q) \vdash \square(p \wedge q)$. But $\square(\square p \wedge \square q) \equiv \square p \wedge \square q$, since $\square p \wedge \square q$, by definition, belongs to Th, and so $\square p \wedge \square q \vdash \square(p \wedge q)$. On the other hand, $\square(p \wedge q) \vdash \square p$ and $\square(p \wedge q) \vdash$

$\Box p$ imply $\Box(p \wedge q) \vdash \Box p \wedge \Box q$, by statement (iv) of Definition 1.3. Thus $\Box(p \wedge q) \equiv \Box p \wedge \Box q$, as desired.

The assertion (C1–C4) follows by duality. $\Box$

We close with a comment on the equivalence of topo-propositional theories. Any map of the set of basic propositions of a theory (L_1, Th_1) into the class of propositions of a theory (L_2, Th_2) evidently extends to a map of all propositions of (L_1, Th_1) to propositions of (L_2, Th_2), and if this extension takes the axioms of the former to provable entailments of the latter we have an interpretation of (L_1, Th_1) in (L_2, Th_2), which amounts to a model of (L_1, Th_1) in $(\Lambda(L_2), \Lambda(Th_2))$, or alternatively a topoframe homomorphism $(\Lambda(L_1), \Lambda(Th_1)) \longrightarrow (\Lambda(L_2), \Lambda(Th_2))$. Clearly, this suggests an obvious notion of equivalence: (L_1, Th_1) is called equivalent to (L_2, Th_2) if the resulting $(\Lambda(L_1), \Lambda(Th_1)) \longrightarrow (\Lambda(L_2), \Lambda(Th_2))$ is an isomorphism. In particular, this situation may arise in the special form where one has a one-one onto map between the basic propositions of (L_1, Th_1) and (L_2, Th_2) providing an interpretation of either in the other.

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(Mohammad Zarghani) Faculty of Mathematics and Computer Sciences, Hakim Sabzevari University, P.O. Box 397, Sabzevar, Iran.[1]

E-mail address: zarghanimohammad@gmail.com

(Ali Akbar Estaji) Faculty of Mathematics and Computer Sciences, Hakim Sabzevari University, P.O. Box 397, Sabzevar, Iran.

E-mail address: aaestaji@hsu.ac.ir

(Abolghasem Karimi Feizabad) Department of Mathematics, Gorgan Branch, Islamic Azad University, Gorgan, Iran.

E-mail address: akarimi@gorganiau.ac.ir

[1]Speaker