SHELAH CARDINALS: A FLEETING GLIMPSE

ALI SADEGH DAGHIGHI [1,2]

Abstract

Shelah cardinals, which lie between Woodin and supercompact cardinals in the consistency strength hierarchy, were originally introduced by Shelah in connection with some problems in measure theory and descriptive set theory.

Definition 0.1. An uncountable cardinal κ is called Shelah, if for every function $f : \kappa \to \kappa$, there exists an elementary embedding $j : V \to M$ with $crit(j) = \kappa$ such that ${}^{\kappa}M \subseteq M$ and $V_{j(f)(\kappa)} \subseteq M$.

Later Gitik and Shelah introduced the generalized notion of an A-Shelah cardinal for a set $A \subseteq {}^{\kappa}\kappa$ as follows:

Definition 0.2. If κ is an uncountable cardinal and A is a set of functions from κ into κ then κ is called A-Shelah, if for every function $f \in A$ there exists an elementary embedding $j : V \to M$ with $crit(j) = \kappa$ such that ${}^{\kappa}M \subseteq M$ and $V_{j(f)(\kappa)} \subseteq M$.

Investigating preservation of large cardinals under various classes of forcing notions started by the work of Levy and Solovay, who showed that measurable cardinals are preserved under small forcing notions. Later works revealed the fact that besides measurable cardinals, the same result holds for a wide range of large cardinals as well.

We prove an analogue of the Levy-Solovay theorem for Shelah cardinals, by showing that Shelah cardinals are preserved under (relatively) small forcings in both upward and downward directions. In other words no Shelah cardinal loses its Shelahness in the generic extension produced by a small forcing and such a forcing doesn't add any new Shelah cardinal to the universe as well.

Theorem 0.3. *Small forcing preserves Shelah cardinals in both upward and downward directions. i.e., If κ is a cardinal and $\mathbb{P}$ is a forcing notion with $|\mathbb{P}| < \kappa$ and $G \subseteq \mathbb{P}$ is a generic filter then κ is Shelah in V if and only if κ is Shelah in $V[G]$.*

Date: Received: , Accepted: .

[1] Department of Mathematics and Computer Science, Amirkabir University of Technology, Hafez avenue 15194, Tehran, Iran. *Email:* a.s.daghighi@gmail.com .

[2] The present lecture is based on a joint work with Massoud Pourmahdian.

We also prove that $({}^{\kappa}\kappa \cap V)$-Shelah cardinals are preserved by Woodin's fast function forcing. As the name suggests, such a forcing adds a new κ-sequence of ordinals $< \kappa$ which in some sense is *faster* than any other such κ-sequence in the ground model and is defined as follows:

Definition 0.4. Let κ be a cardinal. The fast function forcing $\mathbb{P}_\kappa$ consists of all partial functions $p : \kappa \to \kappa$ such that:

(1) $\forall \gamma \in dom(p) \quad p[\gamma] \subseteq \gamma$,
(2) $\forall \gamma \leq \kappa$, if γ is inaccessible then $|dom(p \restriction_\gamma)| < \gamma$.

$\mathbb{P}_\kappa$ is ordered by reverse inclusion.

Theorem 0.5. *Fast function forcing preserves* $({}^{\kappa}\kappa \cap V)$ *- Shelah cardinals.*

Aftar that we deal with the Laver Diamond Principle, isolated by Hamkins as a new combinatorial axiom generalizing the classical Diamond Principle, and prove that such a principle holds for Shelah cardinals. Similar results are already obtained by Laver and Gitik and Shelah for the other large cardinals including supercompact and strong cardinals respectively.

Definition 0.6. For a Shelah cardinal κ let $wt(\kappa)$ be the least ordinal λ such that for any function $f : \kappa \to \kappa$ there exists an extender $E \in V_\lambda$ which witnesses the Shelahness of κ with respect to f. We call $wt(\kappa)$ the witnessing number of κ.

Definition 0.7. $\diamond_\kappa^{Shelah}$ is the assertion: there exists $l : \kappa \to V_\kappa$ such that for any $f : \kappa \to \kappa$ and any $\alpha < wt(\kappa)$ and $A \in V_\alpha$, there are $g > f$ and $j : V \to M$ witnessing the Shelahness of κ with respect to g such that $j(g)(\kappa) > \alpha$ and $j(l)(\kappa) = A$.

Theorem 0.8. *If κ is a Shelah cardinal then $\diamond_\kappa^{Shelah}$ holds.*

Then we use the already obtained Laver function for Shelah cardinals to prove an analogue of Laver's indestructibility of supercompactness theorem for Shelah cardinals by proving that Shelah cardinals can be made indestructible under $\leq \kappa$-directed closed forcings of size $< wt(\kappa)$.

Theorem 0.9. *If κ is Shelah and GCH holds then there is a set forcing extension in which the Shelahness of κ becomes indestructible by any $\leq \kappa$-directed closed forcing of size $< wt(\kappa)$.*

References

[1] A. S. Daghighi and M. Pourmahdian, On Some Properties of Shelah Cardinals, *Bulletin of the Iranian Mathematical Society* (2017), Accepted.