Gödel Incompleteness Theorems, Generalized and Optimized for Definable Theories

Payam Seraji [*](joint work with S. Salehi[†])

Usually, proofs of the Gödel's first incompleteness theorem are stated for recursively enumerable (r.e) theories, but it is known (and a kind of folklore) that this theorem also holds for definable theories (a theory T is called *definable* if the set of (Gödel numbers of) its axioms is definable by a formula $\varphi(x)$, i.e. i is (Gödel number of) an axiom of T, iff $\mathbb{N} \vDash \varphi(\bar{i})$), see e.g. [6] and [5]. Arguments presented in these references need the theory to be sound. We reduce this requirement to Σ_{n-1}-soundness:

Theorem 1. *If (the set of axioms of) a theory $T \supseteq \mathsf{Q}$ is definable by a Σ_n formula, then there is a Π_n sentence independent from T, provided that T is $\Sigma_{n-1}-$sound (which mean T dose not prove any false Σ_{n-1} sentence).*

We also show that this requirement is optimal in a sense, by constructing a Σ_{n-2}-sound and Σ_n-definable extention of Q which is complete. Another kind of consistency statements are the *n-consistency* statements, first introduced by G. Kreisel ([2]). It will be showed that similar incompleteness phenomenon holds for these statements:

Theorem 2. *For any $n \geq 1$, if $T \supseteq Q$ is a $(n-1)$-consistent theory such that the set of its axioms is Σ_n-definable, then there is a Π_n sentence not decidable in T.*

*University of Tabriz.

†University of Tabriz.

This result is optimal too and $(n-1)$-consistency can not be reduced to $(n-2)$-consistency. This theorem is a stronger form of a result of P. Hájek ([1], theorem 2.5). Unfortunately, the proof of this theorem is not constructive and our next theorem shows that it is not an accident, because it is impossible to present a constructive proof for it when $n > 3$:

Theorem 3. *There is no computable function f such that for any Σ_4-definable and 3-consistent theory $T_\psi \supseteq Q$, $f(\ulcorner\psi\urcorner)\downarrow$ and is a Π_4 sentence which is independent from T_ψ (ψ is the formula which defines the set of axioms of T).*

Our last results are about the generalizing of the second incompleteness theorem. the main result is the following:

Theorem 4. *For any Σ_n-definable and Σ_{n-1}-sound theory T extending $I\Delta_0 + Exp$, we have $T \nvdash \Sigma_{n-1}\text{-Sound}(T)$, where $\Sigma_{n-1}\text{-Sound}(T)$ is a sentence that expresses Σ_{n-1} soundness of T.*

We show optimality of this theorem by constructing a Σ_n-definable and Σ_{n-2}-sound theory $\mathcal{T}_n \supseteq PA$ such that $\mathcal{T}_n \vdash \Sigma_{n-2} - sound(\mathcal{T}_n)$ (for any $n \geq 2$).

References

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