Localizing Finite-Depth Kripke Models

(Extended Abstract)

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We can look at a first-order (or propositional) intuitionistic Kripke model as an ordered set of classical models. We showed that for a finite-depth Kripke model in an arbitrary first-order language or propositional language, local (classical) truth of a formula is equivalent to non-classical truth (truth in the Kripke semantics) of a Friedman's translation of that formula, i.e. $\alpha \Vdash A^\rho \Leftrightarrow \mathfrak{M}_\alpha \models A$. We introduce some applications of this fact. We extend the result of [AH02] and show that semi-narrow Kripke models of Heyting Arithmetic HA are locally PA.

D. van Dalen et al. in [vDMKV86] introduced a very useful technique, called *pruning* of a Kripke model, for studying Kripke semantics of HA. Their method is a correspondence between forcing of Friedman's translation of a sentence in a Kripke model, and forcing of that sentence in a sub-model (in the sense of [Vis02]) of the same Kripke model. By this method, they proved that every finite Kripke model of HA is PA-normal, and every ω-frame Kripke model of HA is locally PA for infinitely many nodes of the model. Then K. F. Wehmeier in [Weh96] strengthened this result to a wider class of Kripke models, e.g., finite-depth Kripke models, and some special infinite Kripke models. Ardeshir and Hesaam in [AH02] showed that every rooted narrow tree-frame Kripke model of HA is locally PA. In this paper, by iterated use of the pruning lemma introduced in [vDMKV86], we show that for any node α of a finite depth Kripke model, there exists a sentence ρ, such that for all formula A

$$\alpha \Vdash A^\rho \text{ if and only if } \alpha \vDash A,$$

where A^ρ is Friedman's translation of A by ρ. More precisely we prove the following theorem:

Theorem 0.1. *Suppose $\mathcal{K} = (K, \leq, D, \Vdash)$ is a finite-depth Kripke model for the language $\mathcal{L}$. Then for any $\alpha \in K$, there exists some $\rho \in \mathsf{PEM}(\mathcal{L})^*$ such that for any sentences A in $\mathcal{L}(D(\alpha))$,*

$$\alpha \Vdash A^\rho \textit{ iff } \alpha \models A.$$

Then we prove another version of the above theorem such that cover some infinite Kripke models:

Theorem 0.2. *For a semi-narrow Kripke model $\mathcal{K} = (K, \leq, D, \Vdash)$ with tree frame for a language $\mathcal{L}$ and any $\alpha \in K$, there exists some $\rho \in \mathsf{PEM}_{\mathsf{sen}}(\mathcal{L}(D(\alpha)))^*$ such that for all sentences $A \in \mathcal{L}(D(\alpha))$,*

$$\alpha \Vdash (A^\forall)^\rho \quad \textit{iff} \quad \alpha \models A$$

Intuitively, a semi-narrow Kripke model, is a Kripke model for which the infinite branches are finite and very similar to ω-shapes. More precisely:

Definition 0.3. *A Kripke model is narrow if there is no infinite set of pairwise incomparable nodes. We say that a Kripke model is semi-narrow, if for any set of pairwise incomparable nodes X there is some n such that for almost all $u \in X$ (all but finitely many of them), we have $d(u) \leq n$.*

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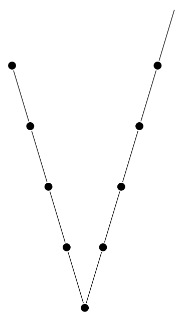
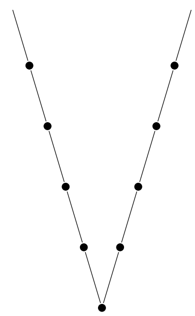
Note that all finite depth Kripke models and also all narrow Kripke models are semi-narrow, but the converse is not necessarily true. For example the comb frame is semi-narrow and it is neither narrow nor finite-depth.



“tick” frame



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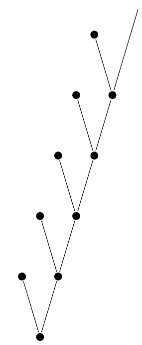

“comb” frame

Then we strengthen Theorem 0.1. We will examine the question whether is it possible to minimize the set PEM in Theorem 0.1? In Theorem 0.4, we show that PEM_1 (see Definition 0.5) is enough, however we do not know if PEM_1 is the minimal set.

Theorem 0.4. *Suppose $\mathcal{K} = (K, \leq, D, \Vdash)$ is a finite-depth Kripke model for the language $\mathcal{L}$. For any $\alpha \in K$, there exists some $\rho \in \mathsf{PEM}_1^*$ such that for any sentence A in $\mathcal{L}(D(\alpha))$,*

$$\alpha \Vdash A^\rho \text{ iff } \alpha \models A.$$

Definition 0.5. *Let $\mathcal{L}$ be an arbitrary first-order language or propositional language. The Kripke-rank of a formula $A \in \mathcal{L}$, $h_{\mathcal{L}}(A)$, is the minimum number n, such that there exists some depth-n Kripke model refusting A, $\mathcal{K} \nVdash A$. If there is some infinite-depth Kripke model which refutes A and no finite-depth Kripke model refuting A, then we define $h(A) := \omega$. If there is no Kripke model $\mathcal{K} \nVdash A$, we define $h(A) := \infty$. For a set of formulas $\Gamma \subseteq \mathcal{L}$, let $\Gamma_n := \{A \in \Gamma : h_{\mathcal{L}}(A) = n\}$.*

References

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