NIP inside a model and Baire 1 definability

Karim Khanaki*
Arak University of Technology

(A collaboration with Anand Pillay†)
University of Notre Dame

September 15, 2017

Abstract

We define the notion $\phi(x, y)$ has NIP in A, where A is a subset of a model, and give some equivalences by translating results from [1]. Using additional material from [11] we discuss the number of coheirs when A is not necessarily countable. We also revisit the notion "$\phi(x, y)$ has NOP in a model M" from [8].

This paper is a kind of companion-piece to [8], although here we are mainly concerned with direct translations of theorems from [1] into the (classical) model theory context. The main results are Corollary 2.2 on equivalences of "$\phi(x, y)$ has NIP in A", Proposition 2.3 on number of coheirs when A is not necessarily countable, and Lemma 2.6 showing that the definition of "$\phi(x, y)$ has not the order property in M" has an equivalent formulation compatible with the NIP definitions in the current paper.

Our model theory notation is standard, and texts such as [9], [7] will be sufficient background for the model theory part of the paper. IP stands for the independence property, and NIP for not the independence property.

*Partially supported by IPM grant 95030059
†Partially supported by NSF grants DMS-1360702 and DMS 1665035

Definition 0.1. *Let T be a complete L-theory, $\phi(x,y)$ an L-formula, and M a model of T.*
(i) A set $\{a_\alpha : \alpha < \kappa\}$ of $l(x)$-tuples from M is said to be an IP-witness for $\phi(x,y)$ if for all finite disjoint subsets I, J of κ, $M \models \exists y(\bigwedge_{\alpha \in I} \phi(a_\alpha, y) \wedge \bigwedge_{\beta \in J} \neg\phi(a_\beta, y))$.
(ii) Let A be a set of $l(x)$-tuples from M. Then $\phi(x,y)$ has IP in A if there is a countably infinite sequence $(a_i : i < \omega)$ of elements of A which is an IP-witness for $\phi(x,y)$.
(iii) Let A be a set of $l(x)$-tuples in M. We say that $\phi(x,y)$ has NIP in A if it does not have IP in A.
(iv) $\phi(x,y)$ has NIP in M if it has NIP in the set of $l(x)$-tuples from M.

Remark 0.2. *(i) ϕ has NIP for the theory T iff it has NIP in every model M of T iff it has NIP in some model M of T in which all types over the empty set in countably many variables are realised.*
(ii) If $\phi(x,y)$ has IP in some model M of T, then there arbitrarily long IP-witnesses for ϕ (of course in different models).
(iii) Let $(a_\alpha : \alpha < \kappa)$ be a collection of $l(x)$-tuples from M and let M^* be a saturated elementary extension of M (i.e. $|M|^+$-saturated). Then $(a_\alpha : \alpha < \kappa)$ is an IP-witness for $\phi(x,y)$ iff there are b_I in M^* for each $I \subseteq \kappa$ such that $M^* \models \phi(a_\alpha, b_I)$ iff $\alpha \in I$, for all $\alpha < \kappa$, $I \subseteq \kappa$.

Given the L-formula $\phi(x,y)$, $\phi^{opp}(y,x)$ is the formula $\phi(x,y)$.

Example 0.3. *Let M be the structure with sorts $P = \omega$, $Q =$ finite subsets of ω, and $R \subset P \times Q$ the membership relation. Then the formula $R(x,y)$ has IP in M whereas $R^{opp}(y,x)$ has NIP in M.*

In [8], the notion "$\phi(x,y)$ has the order property (OP) in a model M" appeared, and we will discuss in 2.3 the compatibilities with Definition 1.1.

It has been known for a long time that the NIP notion arose independently in model theory ([10]) and learning theory [12]. More recently it was noticed by several people (for example [2], [3] and [4]) that the notion also appeared independently in the context of function spaces [1]. The latter paper [1] was at a fairly high level of generality due to trying to find a common context for functions on compact (Hausdorff) spaces and functions on Polish spaces. The compact space case suffices for our purposes.

The current paper is partly expository, and has thematic overlap with [11], [4], [3]. The notion "NIP of $\phi(x, y)$ in a model" is mentioned in [4]. But [11] and [3] deal with "NIP of $\phi(x, y)$ in a theory" (in the continuous framework in the latter paper). In the current paper we work in classical ($\{0, 1\}$-valued) model theory, although results such as Corollary 2.2 are valid in the continuous logic framework. We should also mention that in the continuous framework the formula $||x + y||$ in the language of Banach spaces is NIP in Tsirelson's space $M_{\mathcal{T}}$. This was shown in [5]. It seems to be open whether the same formula has NIP in the theory of the Tsirelson space.

References

[1] J. Bourgain, D.H. Fremlin, M. Talagrand, Pointwise compact sets of Baire-measurable functions, Am. J. Math., vol 100 (1978), 845-886.

[2] A. Chernikov and P. Simon, Definably amenable NIP groups, to appear.

[3] T. Ibarlucía, The dynamical hierachy for Roelcke precompact Polish groups, Israel J. of Math., vol. 215 (2016), no. 2, pp 965-1009.

[4] K. Khanaki, Stability, NIP, and NSOP; Model Theoretic Properties of Formulas via Topological Properties of Function Spaces, ArXiv 2015.

[5] K. Khanaki, $\aleph_0$-categorical Banach spaces contain ℓ_p or c_0, arXiv 2016.

[6] A. Pillay, Dimension theory and homogeneity for elementary extensions of a model, JSL, vol 47 (1982), 147-160.

[7] A. Pillay, *An introduction to stability theory*, OUP, 1983.

[8] A. Pillay, Generic stability and Grothendieck, South American Journal of Logic Vol. 2, n. 2,(2016), p. 1-6.

[9] B. Poizat, *A course in model theory*, Springer, 2000.

[10] S. Shelah, *Classification Theory and the number of nonisomorphic models*, 2nd edition, North Holland, 1990.

[11] P. Simon, Rosenthal compacta and NIP formulas, Fund. Math. vol. 231 (2015), 81-92.

[12] V. Vapnik and A. Chervonenkis, A., On the Uniform Convergence of Relative Frequencies of Events to Their Probabilities, Theory Probab. Appl., 16(2), (2004), 264280.