A Non-Self-Referential Version of Brandenburger-Keisler Paradox

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Abstract

In 2006, Adam Brandenburger and H. Jerome Keisler presented a two-person, self-referential paradox in epistemic game theory which shows the impossibility of completely modeling of players' epistemic beliefs and assumptions. They show that the following configurations of beliefs can not be represented: *Ann believes that Bob assumes that Ann believes that Bob's assumption is wrong.* In this paper, we present a non-self-referential and Yablo–like version of the Brandenburger-Keisler's paradox.

Keywords: Yablo's Paradox · Brandenburger-Keisler Paradox · Epistemic Logic.

1 Introduction

The Brandenburger-Keisler paradox (BK paradox) is a two-person, self-referential paradox in epistemic game theory which shows impossibility of completely modeling of players' epistemic beliefs and assumptions. Brandenburger and Keisler present a modal logic interpretation of the paradox [4]. They introduce two modal operators intended to represent the agents' beliefs and assumptions. In [11], Pacuit approaches the Brandenburger-Keisler paradox from a modal logical perspective and presents a detailed investigation of the paradox in neighborhood models and in hybrid systems. In particular, he shows that the paradox can be seen as a theorem of an appropriate hybrid logic.

The Brandenburger-Keisler paradox is essentially a self-referential paradox and similarly to any other paradox of the same kind can be analyzed from a category theoretical or algebraic point of view. In [1], Abramsky and

Svesper analyze the Brandenburger-Keisler paradox in categorical context. They present the BK paradox as a fixed-point theorem, which can be carried out in any regular category, and show how it can be reduced to a relational form of the one-person diagonal argument due to Lawvere [10] where he gave a simple form of the (one-person) diagonal argument as a fixed-point lemma in a very general setting.

Başkent in [2] approaches the BK paradox from two different perspectives: non-well-founded set theory and paraconsistent logic. He shows that the paradox persists in both frameworks for category theoretical reasons, but, with different properties. Başkent makes the connection between self-referentiality and paraconsistency.

On the other hand, Stephen Yablo introduced a logical paradox in 1993 that is similar to the liar paradox [15], where he used an infinite sequence of statements. Every statement in the sequence refers to the truth values of the later statements. Therefore, it seems this paradox avoids self–reference. In this paper, we apply Yablo's reasoning in his paradox to present a non-self-referential, multi-agent and Yablo–like version of the Brandenburger-Keisler paradox.

2 Brandenburger-Keisler Paradox

Brandenburger and Keisler introduce the following two person Russell-style paradox [2,4,11]. Beliefs and assumptions are two main concepts involved in the statement of the paradox. An assumption is the strongest belief. Suppose there are two players, Ann and Bob, and consider the following description of beliefs:

Ann believes that Bob assumes that Ann believes
that Bob's assumption is wrong.

A paradox arises when one asks the question "Does Ann believe that Bob's assumption is wrong?" Suppose that answer to the above question is "yes". Then according to Ann, Bob's assumption is wrong. But, according to Ann, Bob's assumption is Ann believes that Bob's assumption is wrong. However, since the answer to the above question is "yes", Ann believes that this assumption is correct. So Ann does not believe that Bob's assumption is wrong. Therefore, the answer to the above question must be "no". Thus, it is not the case that Ann believes that Bob's assumption is wrong. Hence Ann believes Bob's assumption is correct. That is, it is correct that Ann believes that Bob's assumption is wrong. So, the answer must have been yes. This is a contradiction!

3 Beliefs and Assumptions in Modal Logic

The models for assumption logic are (Kripke) frames $\mathcal{W} = (W, P)$ where P is a binary relation on W. The elements of W are called worlds, and P is called the accessibility relation. At a world w, $\Box\varphi$ is interpreted as "w believes φ", $\Diamond$ as "w assumes φ", and $A\varphi$ as $\forall z\ \varphi$. The formulas of assumption logic are built from a set L of proposition symbols and the false formula $\bot$ using propositional connectives and the three modal operators $\Box$, $\Diamond$ and A.

In a frame $\mathcal{W}$, a valuation is a function V which associates a subset of $V(\mathbf{D}) \subseteq W$ with each proposition symbol $\mathbf{D} \in L$. For a given valuation V, the notion of a formula φ being **true** at a world w, in symbols $w \models \varphi$, is defined by induction on the complexity of φ. For a proposition symbol $\mathbf{D}$, $w \models \mathbf{D}$ if $w \in V(\mathbf{D})$. The rules for connectives are as usual, and the rules for the modal operators are as follows:

- $w \models \Box\varphi$ if for all $z \in W,\ P(w, z)$ implies $z \models \varphi$.
- $w \models \Diamond\varphi$ if for all $z \in W,\ P(w, z)$ if and only if $z \models \varphi$.
- $w \models A\varphi$ if for all $z \in W,\ z \models \varphi$.

A formula is *valid* for V in $\mathcal{W}$ if it is true at all $w \in W$, and *satisfiable* for V in $\mathcal{W}$ if it is true at some $w \in W$.

Two-player Brandenburger-Keisler paradox can be reformulated to an impossibility result in a modal logic setting [4, 11]. For each pair of players cd among Ann and Bob, there will be an operator $\Box^{cd}$ of beliefs for c about d, and an operator $\Diamond^{cd}$ of assumptions for c about d.

4 Yablo's Paradox

To counter a general belief that all the paradoxes stem from a kind of circularity (or involve some self–reference, or use a diagonal argument) Stephen Yablo designed a paradox in 1993 that seemingly avoided self–reference [14, 15]. Since then much debate has been sparked in the philosophical community as to whether Yablo's Paradox is really circular–free or involves some circularity (at least hidden or implicitly); see e.g. [3, 5–9, 12, 13, 16]. Unlike the liar paradox, which uses a single sentence, this paradox applies an infinite sequence of statements. There is no consistent way to assign truth values to all the statements, although no statement directly refers to itself. Yablo considers the following sequence of sentences $\{S_i\}$:

$$S_1 : \forall k > 1;\; S_k \;\text{ is untrue},$$
$$S_2 : \forall k > 2;\; S_k \;\text{ is untrue},$$
$$S_3 : \forall k > 3;\; S_k \;\text{ is untrue},$$
$$\vdots$$

The paradox follows from the following deductions. Suppose S_1 is true. Then for any $k > 1$, S_k is not true. Specially, S_2 is not true. Also, S_k is not true for any $k > 2$. But this is exactly what S_2 says, hence S_2 is true after all. Contradiction! Suppose then that S_1 is false. This means that there is a $k > 1$ such that S_k is true. But we can repeat the reasoning, this time with respect to S_k and reach a contradiction again. No matter whether we assume S_1 to be true or false, we reach a contradiction. Hence the paradox. Yablo's paradox can be viewed as a non-self-referential liar's paradox.

5 Yablo-like Brandenburger-Keisler Paradox

In this section, we present a non-self-referential version of Brandenburger-Keisler paradox using Yablo's reasoning. Let us consider two infinite sequence of players $\{A_i\}$ and $\{B_i\}$, and following description of beliefs:

$$\begin{array}{cc} A_1 & B_1 \\ A_2 & B_2 \\ A_3 & B_3 \\ \vdots & \vdots \end{array}$$

For all i, A_i believes that B_i assumes that
for all $j > i$, A_j believes that B_j's assumption is wrong.

A paradox arises when one asks the question "Does A_1 believe that B_1's assumption is wrong?"

Suppose that the answer to the above question is "no". Thus, it is not the case that A_1 believes that B_1's assumption is wrong. Hence A_1 believes B_1's assumption is correct. That is, it is correct that for all $j > 1$, A_j believes that B_j's assumption is wrong. Specially, A_2 believes that B_2's assumption is wrong. On the other hand, since for all $j > 2$, A_j believes that B_j's assumption is wrong, one can conclude that A_2 believes B_2's assumption is correct. Therefore, in the same time A_2 believes that B_2's assumption both correct and wrong. This is a contradiction!

If the answer to the above question is "yes". Then according to A_1, B_1's assumption is wrong. But, according to A_1, B_1's assumption is for all $j > 1$,

A_j believes that B_j's assumption is wrong. Thus, there is $k > 1$ for which A_k believes that B_k's assumption is correct. Now we can apply the same reasoning we used before about A_k and B_k to reach the contradiction! Hence the paradox. This paradox is a non-self-referential multi-agent version of the Brandenburger-Keisler paradox.

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