THE PRINCIPLE OF OPEN INDUCTION AND SPECKER SEQUENCES

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Abstract. The schema **ED** asserts that "there exists an intuitionistically enumerable subset of $\mathbb{N}$ which is not intuitionistically decidable". We prove that in the presence of Markov's Principle over Bishop's constructive analysis, $\neg$**ED** is equivalent to the Principle of Open Induction on $[0, 1]$, via Specker sequences.

1. Introduction

The Principle of Open Induction on $[0, 1]$, **OI**$([0, 1])$, which is a consequence of the Principle of Bar Induction, is given by the following statement:

> Let A be an open subset of $[0, 1]$. If A is progressive in $[0, 1]$, then $[0, 1] \subseteq A$, where a subset A of $[0, 1]$ is called progressive in $[0, 1]$ if
>
> $$\forall x \in [0, 1](\forall y \in [0, 1](y < x \to y \in A) \to x \in A).$$

Since A is progressive, $0 \in A$, and since A is an open subset of $[0, 1]$, there is a rational r_0 such that $[0, r_0) \subseteq A$. Again, since A is progressive, $r_0 \in A$, and since A is an open subset of $[0, 1]$, there is a rational r such that $(r_0 - r, r_0 + r) \subseteq A$. Again, since A is progressive, $r_1 := r_0 + r \in A$. We can continue this process indefinitely. The principle of open induction states that we will finally obtain the conclusion $1 \in A$.

Note that classically, we will reach a limit point, like r_ω, by the classically valid fact that a bounded monotone sequence $\langle r_n \rangle$ converges. It is easily can be shown that $r_\omega \in A$. If r_ω is the last point 1, we are done, otherwise we start again, and we reach a second limit point, like $r_{\omega \cdot 2} \in A$. If it isn't again the last point, we continue this process, …, and one would have a series of (open) intervals having for endpoints

$$r_0, r_1, \ldots, r_\omega, \ldots, r_{\omega\cdot 2}, \ldots, r_{\omega^2}, \ldots, r_{\omega^\omega}, \ldots.$$

This is a contradiction, since the above set is uncountable. This is essentially E. Borel's proof of the compactness of a closed interval.

OI was introduced by T. Coquand in a constructive framework and then W. Veldman in [2] provided a list of important equivalent statements of **OI**$([0, 1])$, and among them, we are interested in the following one:

every enumerable subset of $\mathbb{N}$ is nearly-decidable, (**ND**)

where a subset A of $\mathbb{N}$ is called nearly-decidable if and only if

$$\neg\neg\exists\alpha \in 2^{\mathbb{N}}\, \forall n\, (n \in A \leftrightarrow \alpha(n) = 1).$$

M. Ardeshir and R. Ramezanian in [1] introduced the schema **ED** which states that

> there exists an intuitionistically enumerable subset of $\mathbb{N}$ which is not intuitionistically decidable,

where a subset A of $\mathbb{N}$ is intuitionistically decidable if and only if there exists α in $2^{\mathbb{N}}$ such that, for every n, $n \in A$ if and only if $\alpha(n) = 1$, i.e.

$$\exists\alpha \in 2^{\mathbb{N}}\ \forall n\ (n \in A \leftrightarrow \alpha(n) = 1),$$

and a subset A of $\mathbb{N}$ is intuitionistically enumerable if and only if there exists β in $(\mathbb{N} \cup \{\perp\})^{\mathbb{N}}$ such that for every n, $n \in A$ if and only if $\exists k(\beta(k) = n)$, i.e.

$$\exists\beta \in (\mathbb{N} \cup \{\perp\})^{\mathbb{N}}\ \forall n\ (n \in A \leftrightarrow \exists k(\beta(k) = n)).$$

It is proved that **ED** is consistent with some certain well-known axioms of intuitionistic analysis, like the Weak Continuity Principle, the Principle of Bar Induction, the Choice Schema, and the Kripke Schema. It is also shown that **ED** is equivalent to the existence of a *Specker sequence*, a bounded monotone sequence of real numbers without a limit. As is known, Church's Thesis permits the existence of a Specker sequence and then implies **ED**.

2. The main result

Veldman proved that, in the presence of Markov's Principle, **ND** is equivalent to **OI**$[0, 1]$, via the Principle of Induction on Enumerable Bars in Baire space $\mathbb{N}^{\mathbb{N}}$ and another principle called **EnDec**?!, stating that:

> Every enumerable subset A of $\mathbb{N}$ with the property that every decidable and proper subset of $\mathbb{N}$ that is a subset of A is a proper subset of A, coincides with $\mathbb{N}$.

One can show that $\neg$**ED** is equivalent to **ND**. We prove that $\neg$**ED** is equivalent to the Principle of Open Induction on $[0, 1]$ via the concept of Specker sequences, in the presence of Markov's Principle:

$$\forall x(A(x) \vee \neg A(x)) \wedge \neg\neg\exists x\, A(x) \to \exists x\, A(x).$$

It also provides an easy access to the topic of open induction, a topic which is gaining attention recently, via relating open induction with Specker sequences.

Our work may be considered as a research on constructive reverse mathematics based on Bishop's constructive mathematics.

References

[1] M. Ardeshir and R. Ramezanian. Decidability and Specker sequences in intuitionistic mathematics. *MLQ Math. Log. Q.* 55 (6), 637-648 (2009).
[2] W. Veldman. The Principle of Open Induction on Cantor space and the Approximate-Fan Theorem. *arXiv:1408.2493v1* (2014).