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On Large Cardinal Strength of Definable Tree Property[1]

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Abstract Strengthening a result of Amir Leshem [5], we prove that the consistency strength of holding GCH together with definable tree property for all successors of regular cardinals is precisely equal to the consistency strength of existence of proper class many Π^1_1 - reflecting cardinals. Moreover it is proved that if κ is a supercompact cardinal and $\lambda > \kappa$ is measurable, then there is a generic extension of the universe in which κ is a strong limit singular cardinal of cofinality ω, $\lambda = \kappa^+$, and the definable tree property holds at κ^+. Additionally we can have $2^\kappa > \kappa^+$, so that SCH fails at κ.

Keywords Aronszajn Tree · Definable Tree Property · Π^1_1 - Reflecting Cardinal · Easton Reverse Iteration · Extender Based Prikry Forcing

Introduction

The tree property for a regular cardinal κ is the statement that there is no κ - Aronszajn tree or equivalently every κ - tree has a cofinal branch. In general constructing a model for tree property on a regular cardinal κ is not trivial

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and needs large cardinal assumptions. The problem becomes even harder and needs stronger large cardinal assumptions when one tries to get tree property on several successive regular cardinals. In this direction we have:

Proposition 1 *The following results are known about tree property:*
(1) (Konig) The tree property holds on $\aleph_0$*.*
(2) (Aronszajn) The tree property does not hold on $\aleph_1$*.*
(3) (Specker) For every infinite cardinal κ *if* $\kappa^{<\kappa} = \kappa$ *then the tree property does not hold on* κ^+*. Specially if* CH *holds then* $\aleph_2$ *does not have the tree property.*
(4) (Silver - Mitchell) The tree property on $\aleph_2$ *is equiconsistent with the existence of a weakly compact cardinal.*
(5) (Abraham) Assuming the consistency of a supercompact cardinal and a weakly compact above it, it is consistent to have tree property on both $\aleph_2$ *and* $\aleph_3$*.*
(6) (Magidor) The consistency of tree property on both $\aleph_2$ *and* $\aleph_3$ *implies the consistency of "*$0^\sharp$ *exists".*
(7) (Cummings - Foremann) Assuming the existence of an ω*-sequence of supercompact cardinals, it is consistent that the tree property holds for all* $\aleph_n$*'s,* $1 < n < \omega$*.*

Proof For (1), (2), (3) see [4]. (4) is proved in [6]. For (5) and (6) see [1]. The result (7) is proved in [3].

An importnat point about the Aronszajn's result in proposition 1 is the essential use of AC in his construction. Thus the existing $\aleph_1$ - Aronszajn tree is *not* definable. Amir Leshem [5] proved that assuming existence of a Π^1_1 - reflecting cardinal, it is consistent that a definable version of tree property (definition 2) holds on $\aleph_1$.

Definition 1 An inaccessible cardinal κ is Π^m_n - reflecting, if for every $A \subseteq V_\kappa$ definable over V_κ with parameters from V_κ and for every Π^m_n - sentence Φ, if $(V_\kappa, \in, A) \models \Phi$ then there is an $\alpha < \kappa$ such that $(V_\alpha, \in, A \cap V_\alpha) \models \Phi$.

Definition 2 Let κ be a regular cardinal. A κ - tree $(T, <_T)$ is definable if its underlying set is κ, and the relation $<_T$ is Σ_n - definable in the structure $(H_\kappa, \in)$ for some natural number n. We say the definable tree property holds on κ if every definable κ - tree has a cofinal branch.

Remark 1 In his paper [5], Leshem considers several variants of definable tree property, including what he calls definable tree property in the strict, wide and very wide sense. His results are about definable tree property in the strict sense which is exactly what we stated in the definition 2. According to Leshem's definitions, every definable κ - tree in the strict sense is definable in the wide sense and every definable κ - tree in the wide sense is definable in the very wide sense. Also every definable κ - tree $(T, <_T)$ in the wide sense is isomorphic to a κ - tree $(\kappa, <^*)$ that is definable in the strict sense. So it follows that without losing generality one can assume that the definable tree property in the strict and wide sense are identical while the definable tree property in the very wide sense is different from them.

Theorem 1 *(Leshem) The following statements are equiconsistent:*

(1) The definable tree property holds on $\aleph_1$.
(2) There is a Π^1_1 *- reflecting cardinal.*

Proof [5].

As the first part of the work, we generalize Leshem's result to the consistency of definable tree property for proper class of all successors of regular cardinals using the existence of proper class many Π^1_1 - reflecting cardinals, a large cardinal assumption weaker than the existence of a Mahlo cardinal and much weaker than what is theoretically expected for achieving tree property in the usual sense for this class of regular cardinals.

Main Theorem 2 *The following statements are equiconsistent:*

(1) The definable tree property on successor of every regular cardinal.
(2) There are proper class many Π^1_1 *- reflecting cardinals.*

The situation for the consistency of holding tree property at successor of a singular cardinal is generally more complicated than the case of regulars. By a result of Magidor and Shelah [7] it is known that if λ is the singular limit of λ^+ - supercompact cardinals then λ^+ has the tree property. This fact is used by them to prove the consistency of tree property on $\aleph_{\omega+1}$ from a very strong large cardinal assumption. Later Sinapova [10] decreased the necessary large cardinal assumption for proving the consistency of tree prperty on $\aleph_{\omega+1}$ to the existence of ω - many supercompact cardinals.

On the other hand, answering an old question of Woodin, Neeman [9] produced, assuming the existence of ω-many supercompact cardinals, a model in which SCH fails at a singular strong limit cardinal κ of cofinality ω and κ^+ has the tree property. But in Neeman's model, GCH fails cofinally often below κ, and it is still an open problem if we can have a singular cardinal κ such that GCH holds below κ, $2^\kappa > \kappa^+$, and κ^+ has the tree property.

Finally we prove the main theorem 3 which gives an affirmative answer to this question if the tree property is replaced with the definable tree property. Our proof also reduces the large cardinal strength from the existence of infinitely many supercompact cardinals to the existence of a supercompact cardinal and a measurable above it.

Main Theorem 3 *Assume* GCH *holds,* κ *is supercompact and* $\lambda > \kappa$ *is measurable. Then there is a generic extension of the universe in which:*

(1) κ *is a strongly limit singular cardinal of cofinality* ω,
(2) No bounded subsets of κ *are added, in particular* GCH *holds below* κ,
(3) $\lambda = \kappa^+$ *and the definable tree property holds at* λ,
(4) $2^\kappa = |j(\lambda)|$, *in particular if (in* V*)* $|j(\lambda)| > \lambda^+$, *then* SCH *fails at* κ.

The generic extension in which the above theorem holds is essentially the extension obtained by supercompact extender based Prikry forcing introduced by Merimovich in [8].

Our results show that the *definable* version of tree property is so different in nature from its original form and needs much weaker large cardinal assumptions for proving its consistency.

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