On the Axiomatization of Intuitionistic Linear Temporal Logic of Dynamical Systems

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Abstract

The logic $\mathsf{ITL}^{\mathsf{c}}$ is a variant of intuitionistic linear temporal logic (intuitionistic LTL) that is interpreted over the class of dynamic topological systems. A *dynamic topological system* is a pair $\langle \mathfrak{X}, f \rangle$ where $\mathfrak{X}$ is a topological space and f is a continuous function on X. If $\mathfrak{X}$ is an Alexandrov space, $\langle \mathfrak{X}, f \rangle$ is called a *dynamic Alexandrov system*.

In this paper, we consider the logic $\mathsf{ITL}^{\mathsf{c}}_{\mathsf{A}}$, i.e. intuitionistic LTL interpreted over the class of dynamic Alexandrov systems. This logic is the same as the logic $\mathsf{ITL}^{\mathsf{e}}$, i.e. intuitionistic LTL interpreted over the class of *dynamic Kripke frames*. We give a Hilbert-style axiomatization of $\mathsf{ITL}^{\mathsf{e}}$ and prove its completeness with respect to the class $\mathcal{K}$ of all dynamic Kripke frames. Moreover, we show that $\mathsf{ITL}^{\mathsf{e}}$ is complete with respect to the class $\mathcal{Q}$ of all dynamic Kripke frames based on the set of rational numbers.

This work is part of my Ph.D thesis under supervision of professor Morteza Moniri. The main reference is [10]. The proofs are omitted.

Keywords: Intuitionistic logic, Linear temporal logic, Dynamic topological systems, Alexandrov spaces, Kripke semantics, Hilbert-style proof system.

1 Introduction

Temporal logic is a modal logic in which the truth of formulas depends on time, and is a well-established and successfully used formal tool for specification and verification of state-based systems, see e.g. [13, 3]. There are

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many different models of time. In Linear temporal logic (LTL) which we work with it in this paper, each time (or state) has an unique successor.

What is intuitionistic LTL? This question dose not necessarily have a unique answer. Over the years, there have been various intuitionistic versions of LTL. The main contributions to this direction include the following:

- Ewald [6] produced the first version of an intuitionistically based temporal logic system with 'past' and 'future' tenses.

- Davies [4] defined a natural deduction system for a constructive LTL with only $\bigcirc$ and implication, and showed that this system corresponds to a type system for binding-time analysis via the Curry-Howard isomorphism. Later, Kojima and Igarashi [12] provided sequent calculus and Kripke semantics (or as they call *functional Kripke frames*) to extend Davies' logic with Boolean operators.

- Davoren [5] introduced topological semantics for intuitionistic temporal logics with 'past' and 'future' operators. Recently, Fernández-Duque [7] introduced the logic $\mathsf{ITL}^{\mathsf{c}}$, an intuitionistic version of LTL interpreted over the class of dynamic topological systems, and proved the decidability of a fragment of this logic with just $\bigcirc$, $\Diamond$ and $\forall$.

- Kamide and Wansing [11] presented a semantical and proof theoretic study of constructive and bounded time versions of LTL with $\bigcirc$, $\Box$.

- Boudou et al. [2] suggested the logic $\mathsf{ITL}^{\mathsf{e}}$, an intuitionistic version of LTL based on expanding frames (in the present paper called *dynamic Kripke frames*) with $\bigcirc$, $\Diamond$, $\Box$, and proved that this logic is decidable by showing that it has the effective finite model property.

On the one hand, following the work of Fernández-Duque [7], we consider the logic $\mathsf{ITL}^{\mathsf{c}}$. If we interpret the language of intuitionistic LTL over the class of dynamic Alexandrov systems then we will have the other variant of intuitionistic LTL, which is denoted by $\mathsf{ITL}^{\mathsf{c}}_{\mathsf{A}}$.

On the other hand, following the work of Boudou et al. [2], we consider the logic $\mathsf{ITL}^{\mathsf{e}}$, i.e. intuitionistic LTL interpreted over the class of dynamic Kripke frames. A *dynamic Kripke frame* is a birelational structure of the form $\langle W, R, f\rangle$, where R is a partial order and f is a R-monotone function. Since these structures are similar to expanding products in modal logic [8], those are called *expanding frames* in [2].

It can be shown that dynamic Kripke frames are dynamic Alexandrov systems, and vice versa. Thus, the logic $\mathsf{ITL}^{\mathsf{c}}_{\mathsf{A}}$ is the same as the logic $\mathsf{ITL}^{\mathsf{e}}$.

As mentioned above, Boudou et al. [2] showed that the satisfiability and validity problems for $\mathsf{ITL}^{\mathsf{e}}$ are decidable and they left open the problem of finding a sound and complete axiomatization for this logic.

In the present paper, we give a Hilbert-style axiomatization of $\mathsf{ITL}^{\mathsf{e}}$ (or $\mathsf{ITL}^{\mathsf{c}}_{\mathsf{A}}$) and prove its completeness with respect to the class $\mathcal{K}$ of all dynamic Kripke frames by using the canonical model method. Moreover, we show that $\mathsf{ITL}^{\mathsf{e}}$ is complete with respect to the class $\mathcal{Q}$ of all dynamic Kripke frames based on the set of rational numbers by using the notion of *bounded morphic image* for dynamic Kripke models.

2 Syntax and semantics

Let $\mathbb{P}$ be a countable set of propositional variables. The language $\mathcal{L}$ of the intuitionistic LTL is inductively defined by the following grammar:

$$\varphi ::= \quad p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \bigcirc\varphi \mid \Diamond\varphi \mid \Box\varphi$$

where $p \in \mathbb{P}$. As usual, we define negation by $\neg\varphi = \varphi \to \bot$; logical equivalence by $\varphi \leftrightarrow \psi = (\varphi \to \psi) \wedge (\psi \to \varphi)$; and the truth constant by $\top = \bot \to \bot$. Moreover, we read the temporal operators $\bigcirc$ as 'next'; $\Diamond$ as 'eventually'; and $\Box$ as 'always.

Definition 2.1. *A* dynamic topological system *is a pair* $\langle \mathfrak{X}, f\rangle$*, where* $\mathfrak{X}$ *is a topological space and* $f : X \to X$ *is a continuous function. We say that* $\langle \mathfrak{X}, f\rangle$ *is a* dynamic Alexandrov system *if* $\mathfrak{X}$ *be an Alexandrov space.*

Definition 2.2. *A* dynamic topological model $\mathfrak{M} = \langle \mathfrak{X}, f, V\rangle$*, is a dynamic topological system that is equipped with a valuation* $V : \mathbb{P} \to \tau$*. This valuation is extended inductively to arbitrary formulas as the following form:*

$$\begin{aligned}
V(\bot) &= \emptyset\\
V(\varphi \wedge \psi) &= V(\varphi) \cap V(\psi)\\
V(\varphi \vee \psi) &= V(\varphi) \cup V(\psi)\\
V(\varphi \to \psi) &= \mathbb{I}\big((X \setminus V(\varphi)) \cup V(\psi)\big)\\
V(\bigcirc\varphi) &= f^{-1}(V(\varphi))\\
V(\Diamond\varphi) &= \bigcup_{n\in\omega} f^{-n}(V(\varphi))\\
V(\Box\varphi) &= \mathbb{I}\big(\bigcap_{n\in\omega} f^{-n}(V(\varphi))\big).
\end{aligned}$$

where $\mathbb{I}$ *is the interior operator of topological space* $\mathfrak{X} = \langle X, \tau\rangle$*.*

The intuitionistic LTL that is obtained from the interpretation of the language $\mathcal{L}$ over the class of dynamic topological systems, is denoted by $\mathsf{ITL^c}$. In the definition (2.2), if we consider a dynamic Alexandrov system instead of a dynamic topological system, then we will have a *dynamic Alexandrov model*. Since in a dynamic Alexandrov model the infinite intersections of open sets are open thus we have

$$V(\Box\varphi) = \bigcap_{n\in\omega} f^{-n}(V(\varphi)).$$

This makes us have another variant of intuitionistic LTL that is different from $\mathsf{ITL^c}$, see [7]. We denote this logic by $\mathsf{ITL^c_A}$.

Definition 2.3. *A* dynamic Kripke frame *is a triple* $\mathfrak{F} = \langle W, R, f\rangle$ *where* W *is a non-empty set of states,* R *is a partial order on* W *and* $f : W \to W$ *is a* R*-monotone function, i.e. if* wRv *then* $f(w)Rf(v)$ *for each* $w, v \in W$*.*

Definition 2.4. *A* dynamic Kripke model $\mathfrak{M} = \langle W, R, f, V\rangle$*, is a dynamic Kripke frame that is equipped with a valuation* V *defined as a function* $V : \mathbb{P} \to P(W)$ *that satisfies the* R*-monotonicity condition, i.e. if* $w \in V(p)$ *and* wRv *then* $v \in V(p)$ *for each* $p \in \mathbb{P}$ *and* $w, v \in W$*.*

Definition 2.5. *Let* $\mathfrak{M} = \langle W, R, f, V\rangle$ *be a dynamic Kripke model and* $w \in W$*. The* satisfiability relation $\vDash$ *for arbitrary formulas is defined inductively as follows:*

$\mathfrak{M}, w \vDash p$ *iff* $w \in V(p)$;

$\mathfrak{M}, w \vDash \varphi \wedge \psi$ *iff* $\mathfrak{M}, w \vDash \varphi$ *and* $\mathfrak{M}, w \vDash \psi$;

$\mathfrak{M}, w \vDash \varphi \vee \psi$ *iff* $\mathfrak{M}, w \vDash \varphi$ *or* $\mathfrak{M}, w \vDash \psi$;

$\mathfrak{M}, w \vDash \varphi \to \psi$ *iff for all* v *such that* wRv, *if* $\mathfrak{M}, v \vDash \varphi$ *then* $\mathfrak{M}, v \vDash \psi$;

$\mathfrak{M}, w \vDash \bigcirc\varphi$ *iff* $\mathfrak{M}, f(w) \vDash \varphi$;

$\mathfrak{M}, w \vDash \Diamond\varphi$ *iff for some* $n \in \omega$, $\mathfrak{M}, f^n(w) \vDash \varphi$;

$\mathfrak{M}, w \vDash \Box\varphi$ *iff for all* $n \in \omega$, $\mathfrak{M}, f^n(w) \vDash \varphi$.

The intuitionistic LTL that is obtained from the interpretation of the language $\mathcal{L}$ over the class $\mathcal{K}$ of dynamic Kripke frames, is denoted by $\mathsf{ITL^e}$.

Remark 2.6. *Dynamic Kripke frames are dynamic Alexandrov systems, and vice versa. Thus the logic* $\mathsf{ITL^c_A}$ *is the same as the logic* $\mathsf{ITL^e}$*.*

Fact 2.7. *The satisfiability relation for the logic* $\mathsf{ITL}^{\mathsf{e}}$ *is R-monotone, i.e. if* $\mathfrak{M}, w \vDash \varphi$ *and* wRv *then* $\mathfrak{M}, v \vDash \varphi$.

Proof. For a proof, see [2], Proposition 1. □

Fact 2.8. *There is a computable function* g *such that for every formula* $\varphi \in \mathcal{L}$, *if* φ *is satisfiable (resp. unsatisfiable) then* φ *is satisfiable (resp. falsifiable) in a model* $\mathfrak{M} = \langle W, R, f, V\rangle$ *such that* $|W| \leq g(|\varphi|)$.

Proof. For a proof, see [2], Theorem 31. □

Fact 2.9. *The logic* $\mathsf{ITL}^{\mathsf{e}}$ *is decidable.*

Proof. For a proof, see [2], Corollary 32. □

3 The Proof system

The Hilbert-style axiomatization of the logic $\mathsf{ITL}^{\mathsf{e}}$ (or $\mathsf{ITL}^{\mathsf{c}}_{\mathsf{A}}$) has the following axiom schemes:

(taut)	All intuitionistic tautologies
(itl1)	$\bigcirc\bot \to \bot$
(itl2)	$\bigcirc(\varphi \vee \psi) \to \bigcirc\varphi \vee \bigcirc\psi$
(itl3)	$\bigcirc(\varphi \to \psi) \to (\bigcirc\varphi \to \bigcirc\psi)$
(itl4)	$\bigcirc\Diamond\varphi \to \Diamond\bigcirc\varphi$
(itl5)	$\varphi \vee \bigcirc\Diamond\varphi \to \Diamond\varphi$
(itl6)	$\Box\bigcirc\varphi \to \bigcirc\Box\varphi$
(itl7)	$\Box\varphi \to \varphi \wedge \bigcirc\Box\varphi$

and the inference rules:

(mp) $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$

(nex) $\dfrac{\varphi}{\bigcirc\varphi}$

(dia) $\dfrac{\varphi^n \to \psi}{\Diamond\varphi \to \psi}$

(ind) $\dfrac{\varphi \to \psi \quad \varphi \to \bigcirc\varphi}{\varphi \to \Box\psi}$

By $\varphi^n \to \psi$, we mean an infinite number of premises $\varphi \to \psi$, $\bigcirc\varphi \to \psi$, $\bigcirc^2\varphi \to \psi$ and so on. So, the expanded form of the rule (dia) is as the following:

$$\frac{\varphi \to \psi \quad \bigcirc\varphi \to \psi \quad \bigcirc^2\varphi \to \psi \quad \ldots}{\Diamond\varphi \to \psi}$$

Definition 3.1. *Let* φ *be a formula of* $\mathcal{L}$ *and* Γ *be a set of formulas. We say that* φ *is* derivable from assumptions Γ (in $\mathsf{ITL^e}$) *and denote it by* $\Gamma \vdash \varphi$*, if for some* $\gamma_1, \ldots, \gamma_n$ *in* Γ*,* $\vdash (\gamma_1 \wedge \cdots \wedge \gamma_n) \to \varphi$ *where the provability relation* $\vdash$ *is defined as usual.*

Theorem 3.2. (Deduction Theorem). *Let* φ *and* ψ *be formulas of* $\mathcal{L}$ *and* Γ *be a set of formulas. We have* $\Gamma, \varphi \vdash \psi$ *iff* $\Gamma \vdash \varphi \to \psi$*.*

Theorem 3.3. (Soundness Theorem). *Let* φ *be a formula of* $\mathcal{L}$*. If* $\vdash \varphi$ *then* φ *is valid in the class* $\mathcal{K}$*.*

4 Completeness

Definition 4.1. *Let* Γ *be a set of formulas of* $\mathcal{L}$*. We say that* Γ *is a* theory *if it contains all the theorems of* $\mathsf{ITL^e}$ *and closed under the inference rules (mp), (nex), (dia) and (ind). Also,* Γ *is* consistent *if* $\Gamma \nvdash \bot$*.*

Definition 4.2. *A theory* Γ *is* prime *if satisfies the following conditions:*

(1) Γ *has the disjunction property, that is if* $\varphi \vee \psi \in \Gamma$ *then* $\varphi \in \Gamma$ *or* $\psi \in \Gamma$*, and*

(2) Γ *has the diamond property, that is if* $\Diamond\varphi \in \Gamma$ *then* $\bigcirc^n \varphi \in \Gamma$ *for some* $n \in \omega$*.*

Definition 4.3. (Canonical Model). *The* canonical model *for* $\mathsf{ITL^e}$ *is the triple* $\mathfrak{M}^c = \langle W^c, R^c, f^c, V^c \rangle$ *where:*

(1) W^c *is the set of all consistent prime theories;*

(2) R^c *is the inclusion relation*;

(3) f^c *is the function definded on* W^c *by* $f^c(\Gamma) = \{\varphi \mid \bigcirc\varphi \in \Gamma\}$;

(4) V^c *is the valuation function definded by* $V^c(p) = \{\Gamma \in W^c \mid p \in \Gamma\}$ *for every* $p \in \mathbb{P}$.

Lemma 4.4. $\mathfrak{M}^c$ *is a dynamic Kripke model.*

Proof. It suffices to show that f^c is a R^c-monotone function on W^c. □

Lemma 4.5. (Truth Lemma). *For* $\varphi \in \mathcal{L}$*,* $\mathfrak{M}^c, \Gamma \vDash \varphi$ *iff* $\varphi \in \Gamma$.

Proof. It is proved by induction on φ. □

Lemma 4.6. (Prime Lemma). *If $\nvdash \varphi$ then there is a consistent prime theory Γ such that $\varphi \notin \Gamma$.*

Proof. This is an analogue of the standard Henkin construction. □

Now, we have the completeness theorem for $\mathsf{ITL^e}$ with respect to the class $\mathcal{K}$ of all dynamic Kripke frames.

Theorem 4.7. (Completeness Theorem). *Let φ be a formula of $\mathcal{L}$. If $\vDash_{\mathcal{K}} \varphi$ then $\vdash \varphi$.*

Corollary 4.8. *The proof system for $\mathsf{ITL^e}$ is complete with respect to the class of all finite rooted dynamic Kripke frames.*

5 Completeness for the class $\mathcal{Q}$

Let $\langle W_1, R_1\rangle$ and $\langle W_2, R_2\rangle$ be two Kripke frames. A *p-morphism* from $\langle W_1, R_1\rangle$ to $\langle W_2, R_2\rangle$ is a function $F : W_1 \to W_2$ which satisfies:

(1) *if* wR_1v *then* $F(w)R_2F(v)$ *for every* $w, v \in W_1$, *and*

(2) *if* $F(w)R_2u$ *then* wR_1v *and* $u = F(v)$ *for some* $v \in W_1$.

Definition 5.1. *Let $\mathfrak{M}_1 = \langle W_1, R_1, f_1, V_1\rangle$ and $\mathfrak{M}_2 = \langle W_2, R_2, f_2, V_2\rangle$ be two dynamic Kripke models. We say that a function F from W_1 onto W_2 is a* bounded morphic image from $\mathfrak{M}_1$ to $\mathfrak{M}_2$ *if it satisfies:*

(1) *F is a p-morphism from $\langle W_1, R_1\rangle$ to $\langle W_2, R_2\rangle$,*

(2) *$F \circ f_1 = f_2 \circ F$, i. e. the following diagram commutes:*

$$\begin{array}{ccc} W_1 & \xrightarrow{F} & W_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ W_1 & \xrightarrow{F} & W_2 \end{array}$$

(3) *$V_1(p) = \{w \in W_1 \mid F(w) \in V_2(p)\}$, for every $p \in \mathbb{P}$.*

Lemma 5.2. *Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two dynamic Kripke models and F be a bounded morphic image from $\mathfrak{M}_1$ to $\mathfrak{M}_2$. Then for every formula $\varphi \in \mathcal{L}$ and $w \in W_1$, $\mathfrak{M}_1, w \vDash \varphi$ iff $\mathfrak{M}_2, F(w) \vDash \varphi$.*

Proof. It is proved by induction on the complexity of formula φ. □

Corollary 5.3. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two classes of dynamic Kripke models such that for every model $\mathfrak{M}_2 \in \mathcal{C}_2$, there is a model $\mathfrak{M}_1 \in \mathcal{C}_1$ and a bounded morphic image from $\mathfrak{M}_1$ to $\mathfrak{M}_2$. We have that, if $\mathsf{ITL}^{\mathsf{e}}$ is complete with respect to the class $\mathcal{C}_2$ then $\mathsf{ITL}^{\mathsf{e}}$ is complete with respect to the class $\mathcal{C}_1$.*

Proof. By Lemma (5.2), the proof is obvious. □

Definition 5.4. *A* dynamic infinite binary tree *is a triple $T_2 = \langle \Sigma^*, \preccurlyeq, f \rangle$ where Σ^* is the set of all finite strings over $\Sigma = \{0, 1\}$, $\preccurlyeq$ is the initial segment relation and $f : \Sigma^* \to \Sigma^*$ is a $\preccurlyeq$-monotone function. We denote by $\mathcal{T}_2$, the class of all dynamic Kripke frames based on the infinite binary tree.*

Fact 5.5. *Any finite rooted Kripke frame $\langle W, R \rangle$ is the image of $\langle \Sigma^*, \preccurlyeq \rangle$ under some p-morphism.*

Proof. For a proof, see [9], Theorem 1. □

Lemma 5.6. $\mathsf{ITL}^{\mathsf{e}}$ *is complete with respect to the class $\mathcal{T}_2$.*

We denote by $\mathcal{Q}$, the class of all dynamic Kripke frames based on the set of rational numbers, $\langle \mathbb{Q}, \leqslant \rangle$.

Fact 5.7. *There is a p-morphism from $\langle \mathbb{Q}, \leqslant \rangle$ onto $\langle \Sigma^*, \preccurlyeq \rangle$.*

Proof. For a proof, see [1], Theorem 2.4. □

Corollary 5.8. $\mathsf{ITL}^{\mathsf{e}}$ *is complete with respect to the class $\mathcal{Q}$.*

Proof. The proof is the same as the proof of Lemma (5.6). □

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