Large cardinals, forcing axioms, and mathematical realisms.

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Zermelo-Fraenkel set theory with the Axiom of Choice, ZFC, is the standard foundation for mathematics. As initially proved by Kurt Goedel in the 1930's with his First Incompleteness Theorem, ZFC is intrinsically incomplete, in the sense that there are statements expressible in the language of set theory which are nevertheless undecided on the basis of ZFC. Even more, every reasonable extension of ZFC remains incomplete in the same way. Later, in 1963, Paul Cohen showed, with the use of the forcing method, the existence of lots of natural mathematical statements which are undecided by ZFC, the most famous of which is Cantor's Continuum Hypothesis (CH). It is therefore natural to search for additional natural axioms which, when added to ZFC, suffice to decide such questions as CH. The need to find such axioms is all the more urgent if we assume a realist standpoint, whereby the cumulative hierarchy of sets describes a uniquely specifiable object. According to such a view, a question such as ``How many real numbers are there?'', which of course CH answers, should have a unique solution in this hierarchy.

Large cardinal axioms form a natural hierarchy of axioms extending ZFC. They indeed tend to build a hierarchy, in the sense that any two of these axioms, A and A', are compatible, and in fact often comparable (i.e., A implies A' or A' implies A). These lie beyond the scope of what ZFC can prove, and in fact they transcend ZFC in that they cannot be proved to be consistent assuming just the consistency of ZFC (in this sense, they are different from axioms such as CH, or its negation, both of which can be proved, by forcing, to be consistent together with ZFC). This is essentially the content of Goedel's Second Incompleteness Theorem. Moreover, it is a remarkable empirical fact that all natural mathematical theories can be interpreted within ZFC+A for some suitable large cardinal axiom A. Unfortunately, despite their realist appeal conferred to them by the above (especially their lying in a natural hierarchy), large cardinal axioms do not settle such statements as CH.

Forcing axioms are another family of axioms, naturally arising from the use of forcing in set theory, which do decide questions such as CH, and others. It has been recently proved that sufficiently strong such axioms do decide a lot of statements pertaining to low levels of the cumulative hierarchy (in which CH can be expressed), and in fact provide complete descriptions of such fragments of the universe modulo forcing. In recent joint work with Matteo Viale, we have shown that there are in fact several such strong forcing axioms, providing incompatible pictures of the low levels of the cumulative hierarchy. Completeness modulo forcing at this level is therefore an insufficient criterion for deciding between these axioms.

I will analyse the consequences of this state of affairs vis a vis several realist conceptions of mathematics.