Cut-free Sequent Calculus For F

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Corsi [Cor87] studied Kripke semantics in which no assumption of *persistency* is made. These sublogics of intuitionistic logic has a more strict implication and negation. Ishigaki and Kashima [IK08] introduced cut-free sequent calculus for these logics. Their proof systems have the serious drawback of using a rule with arbitrary number of premises.
In this paper we introduce a cut-free sequent calculus for logic F axiomatized by Corsi.

The system F Let $\mathcal{L} = \{\bot, \wedge, \vee, \to\}$ and $\mathcal{P}$ be a countable set of propositional variables. Formulas of $\mathcal{L}$ is defined as usual. We denote the set of all formulas in $\mathcal{L}$ by $\mathcal{F}$. We use $A, B, C, \ldots$ as metavariable for formulas. The system F is axiomatized as follows:
Axioms:

A1 $A \to A$

A2 $(A \to B) \wedge (B \to C) \to (A \to C)$

A3 $(A \wedge B) \to A$

A4 $(A \wedge B) \to B$

A5 $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$

A6 $A \to A \vee B$

A7 $B \to A \vee B$

A8 $(A \to C) \wedge (B \to C) \to (A \vee B \ \to C)$

A9 $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$

A10 $\bot \to A$

Rules:

$$\mathsf{MP_c}\ \frac{\vdash_{\mathsf{F}} A_1 \qquad \cdots \qquad \vdash_{\mathsf{F}} A_n \qquad \vdash_{\mathsf{F}} A_1 \wedge A_2 \wedge \cdots \wedge A_n \to B}{\vdash_{\mathsf{F}} B}$$

$$\mathsf{AF}\ \frac{\vdash_{\mathsf{F}} B}{\vdash_{\mathsf{F}} A \to B}$$

Corsi proved soundness and weak completeness of F w.r.t the class of all intuitionistic Kripke frames which has no persistency assumption and showed that its modal counterpart is K.

Definition 1. $\vdash_{\mathsf{F}} \Gamma \Rightarrow A$ if there exist a derivation of A from Γ and theorems of F by means of $\mathsf{MP_c}$ (only if $\vdash_{\mathsf{F}} A_1 \wedge \cdots \wedge A_n \to B$

De Jongh and Shirmohammadzadeh [dJM15] proved weak deduction theorem for F.

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In order to introduce a sequent calculus for F, we extend our formulas by a new implication which will work as intuitionistic implication. Let $\mathcal{F}^{\supset} := \mathcal{F} \cup \{A \supset B \mid A, B \in \mathcal{F}\}$. Note that we have not altered the language. $A \supset B$ is an auxiliary expression. We do not allow nesting of $\supset$. A Sequent is any expression of form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multiset of expressions in $\mathcal{F}^{\supset}$ and Δ has at most one element. We call Γ strict implicational if every A in Γ is of form $B \to C$. Here we present the calculus $\mathsf{GF}^{\supset}$.

The Calculus $\mathsf{GF}^{\supset}$

Axioms:

$$id : A \Rightarrow A \qquad \bot : \bot \Rightarrow$$

Structural Rules:

$$\frac{\Gamma \Rightarrow C}{\Gamma, A \Rightarrow C}\ (w \Rightarrow) \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}\ (\Rightarrow w)$$

$$\frac{\Gamma, A, A \Rightarrow C}{\Gamma, A \Rightarrow C}\ (c \Rightarrow)$$

Cut Rule

$$\frac{\Gamma \Rightarrow A \qquad A, \Gamma' \Rightarrow C}{\Gamma, \Gamma' \Rightarrow C}\ (cut)$$

Logical Rules:

$$(\vee \Rightarrow)\ \frac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C} \qquad (\Rightarrow \vee)_i\ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2}\ (i = 1, 2)$$

$$(\wedge \Rightarrow)_i\ \frac{\Gamma, A_i \Rightarrow C}{\Gamma, A_1 \wedge A_2 \Rightarrow C}\ (i = 1, 2) \qquad (\Rightarrow \wedge)\ \frac{\Gamma \Rightarrow A_1 \qquad \Gamma \Rightarrow A_2}{\Gamma \Rightarrow A_1 \wedge A_2}$$

$$(\supset\Rightarrow)\ \frac{\Gamma \Rightarrow A \qquad \Gamma, B \Rightarrow C}{\Gamma, A \supset B \Rightarrow C} \qquad (\Rightarrow\supset)\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \supset B}$$

$$(\Rightarrow\to)\ \frac{A, \Gamma^{\supset} \Rightarrow B}{\Gamma \Rightarrow A \to B}\ \dagger$$

† Γ must be strict implicational.

We have proved the following admissibility results for $\mathsf{GF}^{\supset}$:

Theorem 1. $\mathsf{GF}^{\supset}$ *enjoys cut elimination theorem.*

Lemma 2. *Let Δ be a multiset of length ≤ 1 of $\mathcal{F}$. If $\vdash_{\mathsf{GF}^{\supset}} \Gamma \Rightarrow C$, then there exists a cut-free and $(\Rightarrow\supset)$-free proof for $\Gamma \Rightarrow C$.*

Let Γ and Δ be finite multiset of elements in $\mathcal{F}$ with $|\Delta| \leq 1$. We define Δ^* as Δ, if $\Delta \neq \varnothing$, and $\bot$ otherwise.

Theorem 3 (MAIN RESULT)**.**

$$\vdash_{\mathsf{GF}^{\supset}} \Gamma \Rightarrow \Delta \qquad \textit{if and only if} \qquad \Gamma \vdash_{\mathsf{F}} \Delta^* \tag{1}$$

References

[Cor87] Giovanna Corsi. Weak logics with strict implication. *Mathematical Logic Quarterly*, 33(5):389–406, 1987.

[dJM15] Dick de Jongh and Fatemeh Shirmohammadzadeh Maleki. Subintuitionistic logics with kripke semantics. In *International Tbilisi Symposium on Logic, Language, and Computation*, pages 333–354. Springer, 2015.

[IK08] Ryo Ishigaki and Ryo Kashima. Sequent calculi for some strict implication logics. *Logic Journal of IGPL*, 16(2):155–174, 2008.