# ON THE INDEPENDENT NATURE OF GLOBAL AND WELL - FOUNDED UNIVERSES OF MATHEMATICS


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Abstract. We consider the role of the foundation axiom and various anti-foundation axioms in connection with the nature and existence of elementary self-embeddings of the mathematical universe.


The *axiom of foundation* (AF) is one of the standard axioms of the Zermelo–Fraenkel (ZFC) axiomatization of set theory, asserting that the set-membership relation $\in$ is well-founded. AF is not an essential axiom to produce necessary tools of ordinary mathematics. It just simplifies the behavior of $\in$ relation on the global universe of mathematical objects $V$ by removing some complictaed creatures of the universe which are known as ill-founded sets. For example the axiom of foundation refutes $x \in x$ for any set $x$; in particular, it rules out the existence of *Quine atoms*, sets $x$ for which $x = \{x\}$. After this refinement in the universe of mathematical objects the foundation axiom restricts the global proper class of all sets (V) to the proper class of von Neumann hirerarchy (WF) generated from $\emptyset$ by iterating power set operator. More generally when $A$ is a transitive class define:

$$\begin{aligned}
\mathrm{WF}_0(A) &= A, \\
\mathrm{WF}_{\alpha+1}(A) &= \mathcal{P}(\mathrm{WF}_\alpha(A)) \quad \text{for every ordinal } \alpha, \\
\mathrm{WF}_\alpha(A) &= \bigcup_{\beta<\lambda} \mathrm{WF}_\beta(A) \quad \text{for limit ordinal } \alpha
\end{aligned}$$


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5. For more details see [DGH14].

Now define $\mathrm{WF}(A) = \bigcup_{\beta \in Ord} \mathrm{WF}_\beta(A)$. For example, the class of well-founded sets is simply $\mathrm{WF}(\emptyset)$. When $A$ is a proper class, note that we take only the sub*sets* of the previous stage. It is well-known that within $\mathrm{ZFC}^{-\mathrm{f}}$ (i.e. ZFC$-$AF) axiom of foundation is equivalent to the assumption $V = \mathrm{WF}$. When $V = \mathrm{WF}(A)$, then we say that the universe is *generated* by $A$, and in some of our arguments below, we shall consider models generated by a class of Quine atoms. We use the notation At to refer to the class of Quine atoms, so that $V = \mathrm{WF}(\mathrm{At})$ just in case the universe is generated by its Quine atoms.

All of the anti-foundational theories we consider here are equiconsistent with ZFC and therefore also with GBC. A convenient tool for showing the consistency of non-well-founded set theories is *Rieger's theorem* [Rie57] (cf. [Acz88]), which shows in $\mathrm{ZFC}^{-\mathrm{f}}$ that if $M$ is a class endowed with a relation $E \subseteq M \times M$ that is extensional, set-like (meaning that the $E$-predecessors of any $a \in M$ form a set) and full (in the sense that every subset of $M$ is the $E$-extension of some $a \in M$), then $\langle M, E\rangle$ satisfies all the axioms of $\mathrm{ZFC}^{-\mathrm{f}}$. If global choice is available, then we may also expand $\langle M, E\rangle$ to a model of global choice. In particular, any full transitive class, such as $\mathrm{WF}(A)$ for any transitive class $A$, is a model of $\mathrm{ZFC}^{-\mathrm{f}}$.

The foundation axiom has many restrictive impacts on the universe of mathematical objects. One of the most important effects of this axiom is increasing the *degree of rigidity* of the universe to maximum. In fact in the presence of AF the universe has no non-trivial elementary self-embedding. On the other hand if we remove AF from the foundation the number of automorphisms and self-elementary embeddings of the universe varies over all possible situations (see main theorem 13).

As an important special case, we prove that all the standard proofs of the well-known Kunen inconsistency [Kun78], the theorem asserting that there is no nontrivial elementary embedding of the set-theoretic universe to itself, make use of the axiom of foundation (see [Kan04, HKP12]), and this use is essential, assuming that ZFC is consistent, because there are models of $\mathrm{ZFC}^{-\mathrm{f}}$ that admit nontrivial elementary self-embeddings and even nontrivial definable automorphisms. Thus in the absence of foundation axiom, it seems possible to avoid Kunen inconsistency upper bound restriction on the tree of large cardinals. The same phenomenon happens when we remove the restrictive axiom of constructibility (i.e. the assumption $V = L$) which is contradictory to large cardinal axiom "$0^\sharp$ *exists*" and then many new large cardinals like measurables, strongly compacts, etc., appear in the universe with a normal acceptable consistency content as well as small large cardinals of constructible universe like inaccessibles, Mahlo, weakly compacts, etc. A very important open question here is whether the extremely flexible nature of the global universe of mathematical objects in the absence of AF, can lead us to the new large cardinal axioms beyond current borders of Kunen inconsistency theorem.

Nevertheless, a fragment of the Kunen inconsistency survives without foundation as the claim in $\mathrm{ZFC}^{-\mathrm{f}}$ that there is no nontrivial elementary self-embedding of the class of well-founded sets WF (theorem 1). Also with several common anti-foundational theories, such as Aczel's anti-foundational theory $\mathrm{ZFC}^{-\mathrm{f}} + \mathrm{AFA}$ and Scott's theory $\mathrm{ZFC}^{-\mathrm{f}} + \mathrm{SAFA}$ one can prove that there are no nontrivial elementary embeddings from the set-theoretic universe to itself (theorems 2 and 3).

On the other hand some of the commonly considered anti-foundational theories, such as the Boffa theory BAFA, prove outright the existence of nontrivial automorphisms of the set-theoretic universe, thereby refuting the Kunen assertion in these theories (theorem 4 and corollary 5). Thus, the resolution of the Kunen inconsistency in set theory without foundation highly depends on the specific nature of one's anti-foundational stance.

In order to prove the main flexibility theorem (theorem 13) we examined all possible situations within different anti-foundation axioms (theorems 6, 7, 8, 9, 10, 11, 12). Also one can prove a generalized result in the sense of having a prescribed group of automorphisms of the universe (theorem 14).

**Theorem 1.** *Work either in* $\mathrm{GBC}^{-\mathrm{f}}$ *or in* $\mathrm{ZFC}^{-\mathrm{f}}$. *Then there is no nontrivial* $\Sigma_1$*-elementary embedding* $j\colon \mathrm{WF} \to \mathrm{WF}$. *In particular, every* $\Sigma_1$*-elementary embedding* $j\colon V \to V$ *fixes every well-founded set:* $j(x) = x$ *for all* $x \in \mathrm{WF}$. *Furthermore, the range* $j[V]$ *of any such embedding is a transitive full class.*

**Theorem 2.** *Let* IE *be the assertion that "Isomorphic sets (with* $\in$ *structure) are equal" then under* $\mathrm{GBC}^{-\mathrm{f}} + \mathrm{IE}$, *there is no nontrivial* $\Sigma_1$*-elementary embedding* $j\colon V \to V$ *of the universe to itself.*

**Theorem 3.** *Under* GBC, $\mathrm{GBC}^{-\mathrm{f}} + \mathrm{AFA}$, $\mathrm{GBC}^{-\mathrm{f}} + \mathrm{SAFA}$, $\mathrm{GBC}^{-\mathrm{f}} + \mathrm{FAFA}$, *and more generally,* $\mathrm{GBC}^{-\mathrm{f}} + \mathrm{AFA}^{\sim}$ *for any regular bisimulation concept* $\sim$, *there is no nontrivial* $\Sigma_1$*-elementary embedding of the universe.*

**Theorem 4.** *Work in* $\mathrm{GBC}^{-\mathrm{f}}$ *or* $\mathrm{ZFC}^{-\mathrm{f}}$ *and assume that the universe is generated by its Quine atoms, so that* $V = \mathrm{WF}(\mathrm{At})$.

(1) *If* $j\colon V \to V$ *is an automorphism of the universe* $V$, *then* $j \restriction \mathrm{At}$ *is a permutation of the class of atoms* At.

(2) *Every permutation* $\sigma\colon \mathrm{At} \to \mathrm{At}$ *of the atoms has a unique extension to an automorphism* $\overline{\sigma}\colon V \to V$ *of the entire universe.*

**Corollary 5.** *Work in* $\mathrm{ZFC}^{-\mathrm{f}}$*. If the universe is generated by its Quine atoms,* $V = \mathrm{WF}(\mathrm{At})$*, and there are at least two Quine atoms, then there is a nontrivial automorphism of* $V$*, definable from parameters.*

**Theorem 6.** *Work in* $\mathrm{GBC}^{-\mathrm{f}}$ *or* $\mathrm{ZFC}^{-\mathrm{f}}$*, and assume the universe is generated by its Quine atoms, so that* $V = \mathrm{WF}(\mathrm{At})$*.*

(1) *If* $j\colon V \to V$ *is a* $\Sigma_1$*-elementary embedding, then* $j \restriction \mathrm{At}$ *is an injection of* At *to* At*, and* $j = \overline{j \restriction \mathrm{At}}$ *in the sense of the following definition by recursion. Every permutaion function* $\sigma : At \to At$ *is extendible to* $\overline{\sigma} : \mathrm{WF}(At) \to \mathrm{WF}(At)$ *such that* $\overline{\sigma}(x) := \sigma(x)$ *if* $x \in At$ *otherwise* $\overline{\sigma}(x) := \overline{\sigma}[x]$*.*
(2) *If* At *is a set, every* $\Sigma_1$*-elementary embedding* $j\colon V \to V$ *is an automorphism.*
(3) *(Assuming global choice.) If the class of Quine atoms* At *is a proper class and* $\sigma\colon \mathrm{At} \to \mathrm{At}$ *is injective, then* $j = \overline{\sigma}\colon V \to V$ *is an elementary embedding, that is,* $\Sigma_n$*-elementary for every particular natural number* $n$*.*

**Theorem 7.** *Work in* $\mathrm{ZFC}^{-\mathrm{f}}$ *and assume that the universe is generated from a proper class of Quine atoms. Then there is an embedding* $j\colon V \to V$*, definable from parameters and* $\Sigma_n$*-elementary for every particular* $n$*, which is not an automorphism.*

**Theorem 8.** $\mathrm{ZFGC}^{-\mathrm{f}} + \mathrm{BAFA}$ *proves that every isomorphism of transitive sets can be extended to an automorphism of the universe. In particular, there exist nontrivial automorphisms.*

**Theorem 9.** *Work in* $\mathrm{ZFGC}^{-\mathrm{f}} + \mathrm{BAFA}$ *or* $\mathrm{GBC}^{-\mathrm{f}} + \mathrm{BAFA}$*. The following are equivalent for any class* $M$*.*

(1) $M = j[V]$ *for some elementary embedding* $j\colon V \to V$*.*
(2) $M = j[V]$ *for some* $\Sigma_1$*-elementary embedding* $j\colon V \to V$*.*
(3) $M$ *is transitive and isomorphic to* $V$*.*
(4) $M$ *is a full transitive model of* BAFA*.*

**Theorem 10.** *In the theory* $\mathrm{ZFGC}^{-\mathrm{f}} + \mathrm{BAFA}$*, there is a definable class elementary embedding* $j\colon V \to V$*, which is not an automorphism.*

**Theorem 11.** $\mathrm{ZFC}^{-\mathrm{f}} + \mathrm{BAFA}$ *proves that there is no nontrivial* $\Delta_0$*-elementary embedding* $j\colon V \to V$ *of the universe with itself that is definable (with set parameters) in the first-order language of set theory.*

**Theorem 12.** *If* ZFC *is consistent, there is a model of* $\mathrm{GB}^{-\mathrm{f}} + \mathrm{AC}$ *having a class* $j\colon V \to V$ *that is an elementary embedding of the universe to itself, but in which no class is a nontrivial automorphism of the universe.*

**Theorem 13.** *There are models of* $\mathrm{ZFC}^{-\mathrm{f}}$ *realizing all four separating refinements of the fact that* $\{\mathrm{id}_V\} \subseteq \mathrm{Aut}(V) \subseteq \mathrm{Eem}(V)$.

(1) $\{\mathrm{id}_V\} = \mathrm{Aut}(V) = \mathrm{Eem}(V)$. *Models of* ZFC *and theories in the theorem 3 have no nontrivial automorphisms or elementary self-embeddings of universe, and indeed no nontrivial* $\Sigma_1$*-elementary self-embeddings.*
(2) $\{\mathrm{id}_V\} \subsetneq \mathrm{Aut}(V) = \mathrm{Eem}(V)$. *If* $V = \mathrm{WF}(A)$ *for a set* $A$ *of at least two Quine atoms, then there are nontrivial automorphisms, but no other nontrivial elementary embeddings.*
(3) $\{\mathrm{id}_V\} = \mathrm{Aut}(V) \subsetneq \mathrm{Eem}(V)$. *The model of theorem 12 has no nontrivial automorphisms, but does have a nontrivial elementary embedding.*
(4) $\{\mathrm{id}_V\} \subsetneq \mathrm{Aut}(V) \subsetneq \mathrm{Eem}(V)$. *If* $V = \mathrm{WF}(A)$ *for a proper class of Quine atoms, then there are nontrivial automorphisms, as well nontrivial elementary self-embeddings which are not automorphism.*

**Theorem 14.** *Assume* $\mathrm{GBC}^{-\mathrm{f}}+\mathrm{BAFA}$. *Then for every group* $G$*, there is a transitive set* $A_G$ *whose automorphism group is isomorphic to* $G$*, and the automorphism group of the corresponding cumulative universe* $\mathrm{WF}(A_G)$ *generated over this set is also isomorphic to* $G$*, in the sense that every automorphism of* $A_G$ *extends to a unique automorphism of* $\mathrm{WF}(A_G)$ *and every automorphism of* $\mathrm{WF}(A_G)$ *arises this way.*

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